Homework Set 1—Classical varieties

The problems on this set deal with varieties over an algebraically closed field k. For these problems you can work with Serre's definition of a classical variety, as we did in the examples at the beginning of the semester.

Alternatively, you can interpret the problems as questions about schemes, working with the scheme $\operatorname{Spec}(\mathcal{O}(X))$ for a classical affine variety X, or a gluing of affine schemes for a general variety covered by affine open subsets. For example, the scheme $\mathbb{P}^n_k$ corresponds to classical projective space $\mathbb{P}^n(k)$.

In class we already mentioned that classical varieties are in a certain sense equivalent to reduced schemes locally of finite type over k. Eventually we will describe the equivalence in full detail. This equivalence implies that a solution to any of the problems for varieties also solves the scheme version, and vice versa. Nevertheless, you may find it instructive to think about how to solve some of the problems explicitly using varieties first, and then schemes. Normally you should find that the solution involves the same steps (constructing various ring homomorphisms, etc.) in either setting. The scheme solution will often be more general: for instance, it may solve a version of the problem in which the schemes involved are defined over any ring R, not just an algebraically closed field k.

Extra challenging problems are marked with (*).

1. In class we saw that the parametrization $t \to (t^2, t^3)$ of the curve $C = V(y^2 - x^3) \subseteq \mathbb{A}^2$ is a morphism $\phi\colon \mathbb{A}^1 \to C$ which is a homeomorphism, but not an isomorphism of varieties. This doesn't rule out the possibility that C is isomorphic to $\mathbb{A}^1$ via some other morphism. Show that in fact C is not isomorphic to $\mathbb{A}^1$, by proving that the subring $k[t^2, t^3]$ of $k[t]$ is not isomorphic as a k-algebra to $k[t]$.

2. Let C be the closure in $\mathbb{P}^2$ of the plane curve $V(x^2+y^2-1)$, that is, $C = V(x^2+y^2-z^2)$ in projective coordinates, where $V(z)$ is the line at infinity, and U_z is identified with the affine (x, y)-plane. Let L_t be the projective closure of the line $V(y - t(x + 1))$, *i.e.*, L_t is any line through $(-1, 0)$ except for the vertical line $x = -1$.

(a) Show that L_t meets C in exactly one point Q_t other than $(-1, 0)$.

(b) Identifying the set of all lines through $(-1, 0)$ with $\mathbb{P}^1$, show that the map sending L_t to Q_t extends uniquely to an isomorphism $\mathbb{P}^1 \to C$. Where does the vertical line $L = V(x + 1)$ go?

(c) Assuming $\operatorname{char}(k) \neq 2$, show that C has two points on the line at infinity $V(z)$, and that these correspond under the isomoprhism in (b) to the lines L_t with $t = \pm\sqrt{-1}$. What happens if $\operatorname{char}(k) = 2$?

(d) Prove that if k is any field, not necessarily algebraically closed, the answer to (a) gives a parametrization of all solutions of the equation $x^2 + y^2 = 1$ in k^2, excluding $(-1, 0)$, by elements $t \in k$ such that $t^2 \neq -1$. Use this to show that every triple of positive integers $a^2 + b^2 = c^2$ with no common factor has the form $(p^2 - q^2, 2pq, p^2 + q^2)$ for integers $p > q > 0$.

3. (a) Show that the set of curves of degree d in in $\mathbb{P}^2$ is naturally parametrized by $\mathbb{P}^n$, where $n = \binom{d+2}{2} - 1$. Let $H \subseteq \mathbb{P}^n \times \mathbb{P}^2$ be the 'tautological family' whose fiber over a point of $\mathbb{P}^n$ is the curve parametrized by that point. Note that if f is a polynomial separately homogeneous in the projective coordinates on $\mathbb{P}^n$ and $\mathbb{P}^2$, then the zero locus $V(f) \subseteq \mathbb{P}^n \times \mathbb{P}^2$ makes sense and is a closed subvariety. Find such an f so that $H = V(f)$.

(b) For $d = 2$, let Y be the locus in $\mathbb{P}^5$ which parametrizes quadratic curves (conics) that degenerate to two lines or a double line. Show that Y is a closed subvariety of $\mathbb{P}^5$ and find its equation(s).

(c*) Repeat (b) for $d = 3$, where $Y \subseteq \mathbb{P}^9$ parametrizes cubic curves that degenerate to a line and a conic, or three lines, or a line and a double line, or a triple line.

Remark: for general d, the degenerate curves, *i.e.*, those that are not reduced and irreducible, are parametrized by a closed subvariety $Y \subseteq \mathbb{P}^n$. This follows from 'elimination theory,' which is another way of saying that it is a corollary to the geometric theorem that *projective morphisms are proper*. We'll eventually prove the latter theorem.

4. We can regard $\mathrm{SL}_2(k)$ as the affine variety $V(ad - bc - 1)$ in the space $\mathbb{A}^4(k)$ of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Verify that the group law in $\mathrm{SL}_2(k)$ and the map sending an element to its inverse are morphisms of algebraic varieties.

(b) Identifying $\mathbb{P}^1(k)$ with the set of one-dimensional subspaces of k^2, the group $\mathrm{SL}_2(k)$ acts on it, via its action on k^2. More explicitly, an element of $\mathrm{SL}_2(k)$ sends $(z : w) \in \mathbb{P}^1$ to $(z' : w')$, where

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

Show that on the affine line $\{(z : 1)\} \subset \mathbb{P}^1$, this action is the fractional linear transformation sending z to $(az + b)/(cz + d)$, for $cz + d \neq 0$.

(c) Show that the action of any fixed matrix in $\mathrm{SL}_2(k)$ is a morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$, by covering $\mathbb{P}^1$ with affine open sets on which it is given by a polynomial map (you will need different open coverings on $\mathbb{P}^1$ regarded as the domain and codomain of the map).

(d*) Show that the group action as a whole is a morphism $\mathrm{SL}_2(k) \times \mathbb{P}^1 \to \mathbb{P}^1$.

(e*) Show that the group $\mathrm{PSL}_2(k) = \mathrm{SL}_2(k)/\{\pm I\}$ is also an affine variety, in such a way that the group law and the action of $\mathrm{PSL}_2(k)$ on $\mathbb{P}^1$ are morphisms. Hint: the coordinate ring of $\mathcal{O}(\mathrm{PSL}_2(k))$ is the subring of $\mathcal{O}(\mathrm{SL}_2(k))$ consisting of functions constant on cosets of $\{\pm I\}$.

5. Prove that every global regular function on projective space $\mathbb{P}^n$ is constant. If you do this using varieties, you should assume that the ring of global regular functions $\mathcal{O}_X(X)$ on a classical affine variety X is equal to the coordinate ring $\mathcal{O}(X)$. This is a corollary to the corresponding theorem for schemes, which we already proved, and the equivalence relating the classical variety X to the scheme $\mathrm{Spec}(\mathcal{O}(X))$.

6. (a) Let X be the hypersurface $X = V(xz - wy) \subseteq \mathbb{A}^4$. Let $U = X_y \cup X_z = X - V(y,z)$. Find a regular function $g \in \mathcal{O}_X(U)$ which cannot be expressed in the form $g = h/f$, where h and f are polynomials in the coordinates w, x, y, z such that $f \neq 0$ on U.

(b) Prove that U is isomorphic to the complement of a line in $\mathbb{A}^3$. In particular, U is not affine. Hint: use the function g as one of the coordinates on $\mathbb{A}^3$.

(c*) The phenomenon in (a) can occur even if X and the open subset U are both affine. To see this, let $X = V(xz - wy, ay^2 + bz^2 - yz) \subset \mathbb{A}^6$ and let $U = X_y \cup X_z$. Prove that in this case, U is isomorphic to the affine variety $V(uy + vz - 1) \subseteq \mathbb{A}^5$, with the open embedding $U \hookrightarrow X$ corresponding to a ring homomorphism $y \mapsto y$, $z \mapsto z$, $x \mapsto gy$, $w \mapsto gz$, $a \mapsto uz$, $b \mapsto vy$, where u, v, y, z, g are the coordinates on $\mathbb{A}^5$.

7. Let $(x_0 : \cdots : x_3)$ be projective coordinates on $\mathbb{P}^3$. Let $U_0, \ldots, U_3$ be the standard affines $U_i = \mathbb{P}^3 - V(x_i)$, with coordinates $\{x_j \mid j \neq i\}$ on U_i given by fixing $x_i = 1$.

(a) Show that there is a projective variety $Z \subseteq \mathbb{P}^3$ such that $Z \cap U_i$ is given by the equations

$$\begin{aligned} x_2 = x_1^2,\ x_3 = x_1^3 &\quad \text{on } U_0 \\ x_0x_2 = 1,\ x_3 = x_2^2 &\quad \text{on } U_1 \\ x_1x_3 = 1,\ x_0 = x_1^2 &\quad \text{on } U_2 \\ x_1 = x_2^2,\ x_0 = x_2^3 &\quad \text{on } U_3. \end{aligned}$$

(b) Find homogeneous equations of Z in projective coordinates.

(c) Construct a morphism $\mathbb{P}^1 \to \mathbb{P}^3$ whose image is Z. Is Z isomorphic to $\mathbb{P}^1$?

8. (a) Prove that for every non-zero linear form $f = a_0x_0 + \cdots + a_nx_n$ in the projective coordinates $(x_0 : \cdots : x_n)$, the open subvariety $U = \mathbb{P}^n - V(f)$ is isomorphic to $\mathbb{A}^n$.

(b*) Prove that for every non-zero homogeneous polynomial $f(x_0, \ldots, x_n)$, the open subvariety $U = \mathbb{P}^n - V(f)$ is affine. Hint: for coordinates on U take all the functions x^m/f, where x^m is a monomial in the x_i of degree $d = \deg(f)$. Then show that U is isomorphic to the affine variety X with coordinate ring $\mathcal{O}(X) = R/(f(x) - 1)$ where $R \subseteq k[x_0, \ldots, x_n]$ is the subalgebra generated by monomials of degree d.

9. We can identify the graph of a polynomial function $f(x_1, \ldots, x_n)$ with the affine variety $X = V(y - f(x)) \subseteq \mathbb{A}^{n+1}$. Prove that $y - f(x)$ generates the ideal $\mathcal{I}(X) \subseteq k[x_1, \ldots, x_n, y]$ and that X is isomorphic to $\mathbb{A}^n$.

10. Let $X = V(y - f(x)) \subseteq \mathbb{A}^{n+1}$ be the graph of f, as in the previous problem. Construct a natural bijective correspondence between ring homomorphisms $\mathcal{O}(X) \to k[t]/(t^2)$ and pairs (p, v), where $p \in X$ and v is a tangent vector to X at p, that is, a vector in k^{n+1} such that the directional derivative of $y - f(x)$ in the v direction vanishes at p. Note that the derivatives of a polynomial make sense formally with coefficients in any field.